An iterative uncoupling technique for the identification of the dynamic properties of joints

F. C. Batista[1], N. M. M. Maia[2]
[1] Polytechnic Institute of Leiria, School of Technology and Management
Morro do Lena, 2401-951 Leiria, Portugal

[2] IDMEC/IST, Tech. Univ. of Lisbon
Av. Rovisco Pais, 1049-001 Lisboa, Portugal
e-mail: **nmaia@dem.utl.ist.pt**

Abstract

The identification of the dynamic properties of joints is a very important subject in structural dynamics. However, classic uncoupling methods are often very difficult to apply. In the present paper an iterative technique is proposed, trying to circumvent the well-known ill-conditioning problems that arise due to any small perturbations existing in the experimental data. Various cases are simulated and discussed to illustrate the technique, including situations where it is not possible to take measurements on the joint itself; the results are compared with the ones obtained with other methods.

1 Introduction

In recent years various works on the identification of substructures have appeared, based on frequency domain techniques, known as "FRF-based substructuring method" (FBS), as referred to by Klerk *et al* [1]. Those techniques are essentially based on the classic method developed by Jetmundsen *et al* [2], who proposed a different formulation for the FRF coupling algorithm, based on a single inverse, and identified the main issues that happen due to matrix ill-conditioning.

Others authors, like Ren [3], identified the dynamic stiffness of a substructure also involving the inverse of a matrix. Liu and Ewins [4] tried to solve that problem using the singular value decomposition (SVD). In order to simplify the identification of the substructure, Ratcliffe and Lieven [5] took advantage on the repetition of some of the entries of the finite element mass and stiffness matrices. Wang and Chuang [6] assumed that the joint to be identified is massless, making the process easier to perform, using some error functions. All the authors need to perform some kind of inversion and use the entire frequency range.

To avoid the matrix inversion problem, Ren and Beards [7] suggested an iterative method to obtain the dynamic stiffness of a joint. However, his technique revealed some issues, due to its high dependency on the initial estimates for each frequency.

Here, the authors propose a different approach to circumvent the matrix inverse problem, also through an iterative technique. One takes into account the necessity of evaluating the behavior of the algorithm along the frequency range of interest.

2 Theoretical formulation

In order to characterize the dynamic behavior of a joint, one shall use the dynamic stiffnesses $\boldsymbol{Z}$ and the *FRFs* $\boldsymbol{H}$ of the various substructures, including the one that represents the joint. Our complex structure, denoted by *C*, is formed by the coupling of two substructures, *A* and *B* (the joint), as in Figure 1. Substructure *A* is assumed to be easy to model numerically, using the finite element method (*FE*); the internal co-ordinates are designated by *i* and the co-ordinates common to substructure *B* are designated by *j*. These latter ones are often difficult to measure. No internal co-ordinates are considered in substructure *B*.

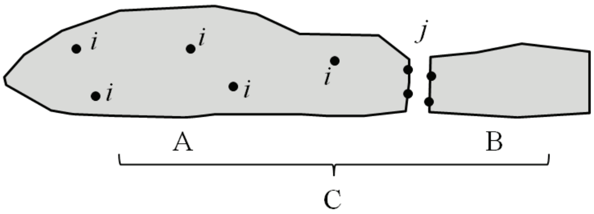

Figure 1 Coupling of substructures *A* and *B*, to form structure *C*.

Let us determine the dynamic stiffness of substructure *B*, $\boldsymbol{Z}^{\boldsymbol{B}}$, using the information obtained numerically from substructure *A* and the measurements taken at co-ordinates *i* of structure *C*.

The dynamic stiffness $\boldsymbol{Z}^{\boldsymbol{C}}$, of structure C, may be written as the summation of the dynamic stiffnesses $\boldsymbol{Z}^{\boldsymbol{A}}$ and $\boldsymbol{Z}^{\boldsymbol{B}}$:

$$\boldsymbol{Z}^{C} = \boldsymbol{Z}^{A} + \boldsymbol{Z}^{B} \tag{1}$$

The r.h.s. of eq. (1) may be recast as:

$$\boldsymbol{Z}^{C} = \boldsymbol{Z}^{A}\left(\boldsymbol{I} + \boldsymbol{H}^{A}\boldsymbol{Z}^{B}\right) \tag{2}$$

Inverting both sides, yields

$$\boldsymbol{H}^{C} = \left(\boldsymbol{I} + \boldsymbol{H}^{A}\boldsymbol{Z}^{B}\right)^{-1}\boldsymbol{H}^{A} \tag{3}$$

Explicitly writing eq. (3) in terms of submatrices involving co-ordinates *i* and *j*, leads to

$$\begin{bmatrix}\boldsymbol{H}_{ii}^{C} & \boldsymbol{H}_{ij}^{C}\\ \boldsymbol{H}_{ji}^{C} & \boldsymbol{H}_{jj}^{C}\end{bmatrix} = \left[\begin{bmatrix}\boldsymbol{I}_{ii} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_{jj}\end{bmatrix} + \begin{bmatrix}\boldsymbol{H}_{ii}^{A} & \boldsymbol{H}_{ij}^{A}\\ \boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A}\end{bmatrix}\begin{bmatrix}\boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{Z}_{jj}^{B}\end{bmatrix}\right]^{-1}\begin{bmatrix}\boldsymbol{H}_{ii}^{A} & \boldsymbol{H}_{ij}^{A}\\ \boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A}\end{bmatrix} \tag{4}$$

which, after simplification, gives:

$$\begin{bmatrix}\boldsymbol{H}_{ii}^{C} & \boldsymbol{H}_{ij}^{C}\\ \boldsymbol{H}_{ji}^{C} & \boldsymbol{H}_{jj}^{C}\end{bmatrix} = \begin{bmatrix}\boldsymbol{I}_{ii} & \boldsymbol{H}_{ij}^{A}\boldsymbol{Z}_{jj}^{B}\\ \boldsymbol{0} & \boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{H}_{ii}^{A} & \boldsymbol{H}_{ij}^{A}\\ \boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A}\end{bmatrix} \tag{5}$$

Further simplification leads to:

$$\begin{bmatrix}\boldsymbol{H}_{ii}^{C} & \boldsymbol{H}_{ij}^{C}\\ \boldsymbol{H}_{ji}^{C} & \boldsymbol{H}_{jj}^{C}\end{bmatrix} = \begin{bmatrix}\boldsymbol{I}_{ii} & -\boldsymbol{H}_{ij}^{A}\boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\\ \boldsymbol{0} & \left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{H}_{ii}^{A} & \boldsymbol{H}_{ij}^{A}\\ \boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A}\end{bmatrix} \tag{6}$$

Thus,

$$\begin{bmatrix} \boldsymbol{H}_{ii}^{C} & \boldsymbol{H}_{ij}^{C} \\ \boldsymbol{H}_{ji}^{C} & \boldsymbol{H}_{jj}^{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_{ii}^{A} - \boldsymbol{H}_{ij}^{A}\boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{ij}^{A} - \boldsymbol{H}_{ij}^{A}\boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{jj}^{A} \\ \left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{ji}^{A} & \left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{jj}^{A} \end{bmatrix} \tag{7}$$

Isolating $\boldsymbol{H}^{A}$, it follows that

$$\begin{bmatrix} \boldsymbol{H}_{ii}^{C} & \boldsymbol{H}_{ij}^{C} \\ \boldsymbol{H}_{ji}^{C} & \boldsymbol{H}_{jj}^{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_{ii}^{A} & \boldsymbol{H}_{ij}^{A} \\ \boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A} \end{bmatrix} - \begin{bmatrix} \boldsymbol{H}_{ij}^{A}\boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{ij}^{A}\boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{jj}^{A} \\ \boldsymbol{H}_{ji}^{A} - \left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A} - \left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{jj}^{A} \end{bmatrix} \tag{8}$$

Both elements of the second row of the second matrix on the r.h.s. of eq. (8) are constituted by differences of matrices. Let us simplify those two elements: from the former, it turns out that

$$\begin{aligned} &\boldsymbol{H}_{ji}^{A} - \left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{ji}^{A} = \left(\boldsymbol{I}_{jj} - \left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\right)\boldsymbol{H}_{ji}^{A} \\ &= \left(\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1} - \left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\right)\boldsymbol{H}_{ji}^{A} \\ &= \left(\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B} - \boldsymbol{I}_{jj}\right)\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\right)\boldsymbol{H}_{ji}^{A} = \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{ji}^{A} \end{aligned} \tag{9}$$

Proceeding in the same way with the latter, one obtains the following result:

$$\begin{bmatrix} \boldsymbol{H}_{ii}^{A} & \boldsymbol{H}_{ij}^{A} \\ \boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A} \end{bmatrix} - \begin{bmatrix} \boldsymbol{H}_{ii}^{C} & \boldsymbol{H}_{ij}^{C} \\ \boldsymbol{H}_{ji}^{C} & \boldsymbol{H}_{jj}^{C} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_{ij}^{A}\boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{ij}^{A}\boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{jj}^{A} \\ \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1}\boldsymbol{H}_{jj}^{A} \end{bmatrix} \tag{10}$$

Pre-multiplying the first element of the first row of eq. (10) by $\left(\boldsymbol{H}_{ij}^{A}\right)^{+}$ and post-multiplying by $\left(\boldsymbol{H}_{ji}^{A}\right)^{+}$, where

$$\begin{aligned} \left(\boldsymbol{H}_{ij}^{A}\right)^{+} &= \left(\boldsymbol{H}_{ji}^{A}\boldsymbol{H}_{ij}^{A}\right)^{-1}\boldsymbol{H}_{ji}^{A} \\ \left(\boldsymbol{H}_{ji}^{A}\right)^{+} &= \boldsymbol{H}_{ij}^{A}\left(\boldsymbol{H}_{ji}^{A}\boldsymbol{H}_{ij}^{A}\right)^{-1} \end{aligned} \tag{11}$$

one obtains:

$$\left(\boldsymbol{H}_{ij}^{A}\right)^{+}\left(\boldsymbol{H}_{ii}^{A} - \boldsymbol{H}_{ii}^{C}\right)\left(\boldsymbol{H}_{ji}^{A}\right)^{+} = \boldsymbol{Z}_{jj}^{B}\left(\boldsymbol{I}_{jj} + \boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B}\right)^{-1} \tag{12}$$

Eq. (12) may be rewritten as:

$$\boldsymbol{Z}_{jj}^{B} = \left(\left(\boldsymbol{H}_{ij}^{A}\right)^{+}\left(\boldsymbol{H}_{ii}^{A} - \boldsymbol{H}_{ii}^{C}\right)\left(\boldsymbol{H}_{ji}^{A}\right)^{+}\right)\boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B} + \left(\boldsymbol{H}_{ij}^{A}\right)^{+}\left(\boldsymbol{H}_{ii}^{A} - \boldsymbol{H}_{ii}^{C}\right)\left(\boldsymbol{H}_{ji}^{A}\right)^{+} \tag{13}$$

Eq. (13) exhibits the "classic" format of the iterative methods for the solution of linear matrix equations [8, 9], whose general form is the following one:

$$\boldsymbol{X}^{(k+1)} = \boldsymbol{G}\boldsymbol{X}^{(k)} + \boldsymbol{P} \tag{14}$$

where $\boldsymbol{G}$ is the iteration matrix. It can be proven [8, 9] that this process converges if and only if the spectral radius of the iteration matrix is less than 1 ($\rho(\boldsymbol{G}) < 1$). The method converges for any initial estimate $\boldsymbol{X}^{(0)}$. Thus, making $\boldsymbol{G} = \boldsymbol{P}\boldsymbol{N}$, one can write eq. (13), for each frequency ω, as

$$\boldsymbol{Z}_{jj}^{B}(\omega)^{(k+1)} = \boldsymbol{P}(\omega) \cdot \boldsymbol{N}(\omega) \cdot \boldsymbol{Z}_{jj}^{B}(\omega)^{(k)} + \boldsymbol{P}(\omega) \tag{15}$$

with

$$\begin{aligned} \boldsymbol{P}(\omega) &= \left(\boldsymbol{H}_{ij}^{A}(\omega)\right)^{+} \left(\boldsymbol{H}_{ii}^{A}(\omega) - \boldsymbol{H}_{ii}^{C}(\omega)\right)\left(\boldsymbol{H}_{ji}^{A}(\omega)\right)^{+} \\ \boldsymbol{N}(\omega) &= \boldsymbol{H}_{jj}^{A}(\omega) \end{aligned} \tag{16}$$

To comply with the convergence requirement ($\rho(\boldsymbol{G}) < 1$), only a limited number of frequencies can be used; the criterion for stopping the process is when the difference between two consecutive solutions is smaller than a certain pre-defined small value δ. As one is dealing with matrices, the Frobenius norm has been elected and therefore the criterion is written as:

$$\left\| \boldsymbol{Z}_{jj}^{B}(\omega)^{(k)} - \boldsymbol{Z}_{jj}^{B}(\omega)^{(k-1)} \right\|_{F} < \delta \tag{17}$$

From eq. (10) one can use the other three elements and proceed in a similar way, ending up with four different formulations to obtain the dynamic stiffness $\boldsymbol{Z}_{jj}^{B}$.

2.1 Summary

The four formulations are:

Formulation 1

$$\boldsymbol{Z}_{jj}^{B} = \left(\left(\boldsymbol{H}_{ij}^{A}\right)^{+} \left(\boldsymbol{H}_{ii}^{A} - \boldsymbol{H}_{ii}^{C}\right)\left(\boldsymbol{H}_{ji}^{A}\right)^{+}\right)\boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B} + \left(\boldsymbol{H}_{ij}^{A}\right)^{+} \left(\boldsymbol{H}_{ii}^{A} - \boldsymbol{H}_{ii}^{C}\right)\left(\boldsymbol{H}_{ji}^{A}\right)^{+} \tag{18}$$

Formulation 2

$$\boldsymbol{Z}_{jj}^{B} = \left(\left(\boldsymbol{H}_{ij}^{A}\right)^{+} \left(\boldsymbol{H}_{ij}^{A} - \boldsymbol{H}_{ij}^{C}\right)\left(\boldsymbol{H}_{jj}^{A}\right)^{-1}\right)\boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B} + \left(\boldsymbol{H}_{ij}^{A}\right)^{+} \left(\boldsymbol{H}_{ij}^{A} - \boldsymbol{H}_{ij}^{C}\right)\left(\boldsymbol{H}_{jj}^{A}\right)^{-1} \tag{19}$$

Formulation 3

$$\boldsymbol{Z}_{jj}^{B} = \left(\left(\boldsymbol{H}_{jj}^{A}\right)^{-1} \left(\boldsymbol{H}_{ji}^{A} - \boldsymbol{H}_{ji}^{C}\right)\left(\boldsymbol{H}_{ji}^{A}\right)^{+}\right)\boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B} + \left(\boldsymbol{H}_{jj}^{A}\right)^{-1} \left(\boldsymbol{H}_{ji}^{A} - \boldsymbol{H}_{ji}^{C}\right)\left(\boldsymbol{H}_{ji}^{A}\right)^{+} \tag{20}$$

Formulation 4

$$\boldsymbol{Z}_{jj}^{B} = \left(\left(\boldsymbol{H}_{jj}^{A}\right)^{-1} \left(\boldsymbol{H}_{jj}^{A} - \boldsymbol{H}_{jj}^{C}\right)\left(\boldsymbol{H}_{jj}^{A}\right)^{-1}\right)\boldsymbol{H}_{jj}^{A}\boldsymbol{Z}_{jj}^{B} + \left(\boldsymbol{H}_{jj}^{A}\right)^{-1} \left(\boldsymbol{H}_{jj}^{A} - \boldsymbol{H}_{jj}^{C}\right)\left(\boldsymbol{H}_{jj}^{A}\right)^{-1} \tag{21}$$

2.2 Simulation study

To assess the performance of these iterative procedures, one shall use a coupling between two beams, connected by a joint element, as in Figure 2. Structure *C* is composed by A_1, *B* and A_2.

The objective is to evaluate the dynamic stiffness of B, $\boldsymbol{Z}_{jj}^{B}$, assuming that $\boldsymbol{H}^{A}$ is calculated analytically and $\boldsymbol{H}^{C}$ is obtained experimentally.

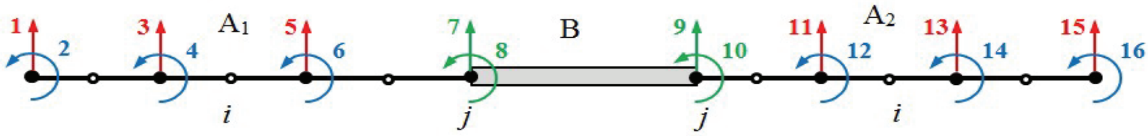

Figure 2: Coupling of substructures A_1, *B* and A_2

Using the finite element method, the Bernoulli-Euler beam element with four degrees of freedom has been chosen. Substructures A_1 and A_2 were discretized into six elements each and only the indicated co-ordinates in Figure 2 have been considered as *i* and *j*, where it is possible to measure and excite the structure. The properties of the beams are shown in Table 1.

Beam	Length	Width	Thickness	E	ρ
A_1	270 mm	30 mm	5 mm	194 GPa	7562 kg/m^3
B	200 mm	30 mm	10 mm	194 GPa	7562 kg/m^3
A_2	370 mm	30 mm	5 mm	194 GPa	7562 kg/m^3

Table 1: Characteristics of the beams

To simulate the experimental errors in $\boldsymbol{H}^{C}$, one shall impose an amplitude independent numerical error, described as [10]:

$$\tilde{H}_{pq}^{C}(\omega_k) = H_{pq}^{C}(\omega_k) + \frac{\gamma}{100} \cdot normrnd(\omega_k) \cdot \max\left(\left|H_{pq}^{C}(\omega)\right|\right) \tag{22}$$

where γ is the noise level (%) and $normrnd(\omega)$ is a *MatLab*® function that represents a normal distribution with zero mean value and standard deviation equal to 1. In this example the noise level will be 3%. The stopping criterion value was $\delta = 10^{-7}$. Once $\boldsymbol{Z}_{jj}^{B}$ is found, a regression using a least squares calculation is performed to find the stiffness and mass associated to each element, which is of the form $k - \omega^2 m$. One of the problems of this process is the shortage of frequencies where the spectral radius of the iteration matrix $\boldsymbol{G}$ is less than 1. This obstacle makes the regression task quite sensitive to any perturbation on the few values that have been used.

Figures 3, 4 and 5 show the results of the estimation of $\boldsymbol{Z}_{11}^{B}$ for formulations 1, 2 and 3. The black curve represents the correct results that should be expected. The points in blue represent the values obtained at each frequency by the iterative method and the red curve represents the regeneration of $\boldsymbol{Z}_{11}^{B}$ after the identification of k and m for each element of the matrix. As it can be observed, the results are not acceptable, as the black and red curves are completely distinct.

However, the results of $\boldsymbol{Z}_{11}^{B}$ for formulation number 4 (see figure 6) are quite reasonable, in spite of considerably high perturbations in some frequency bands.

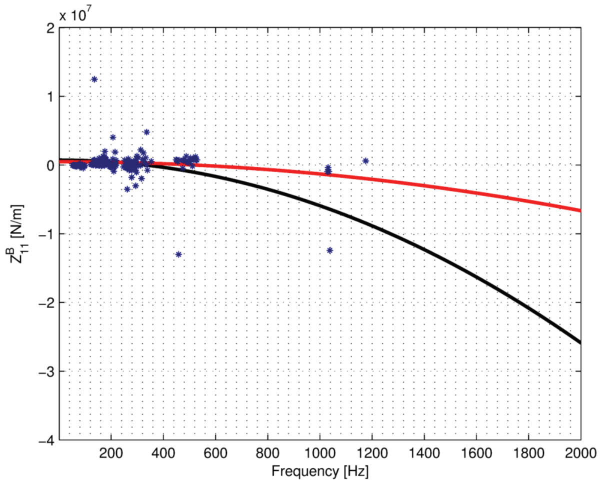

Figure 3: Z_{11}^{B} from formulation 1

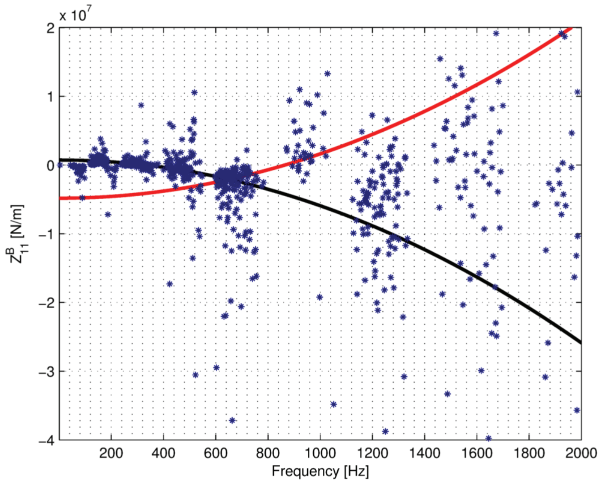

Figure 4: Z_{11}^{B} from formulation 2

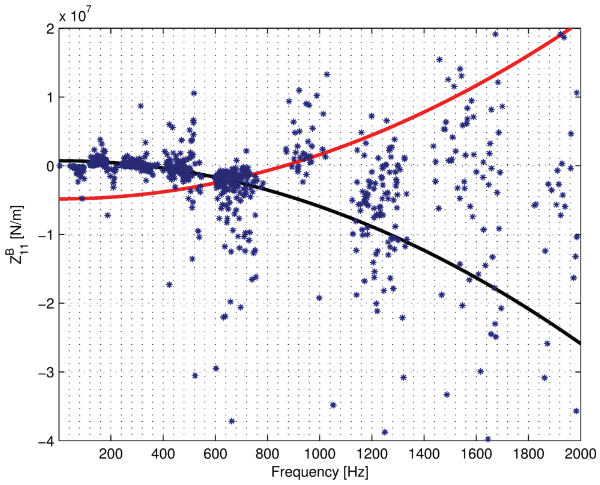

Figure 5: Z_{11}^{B} from formulation 3

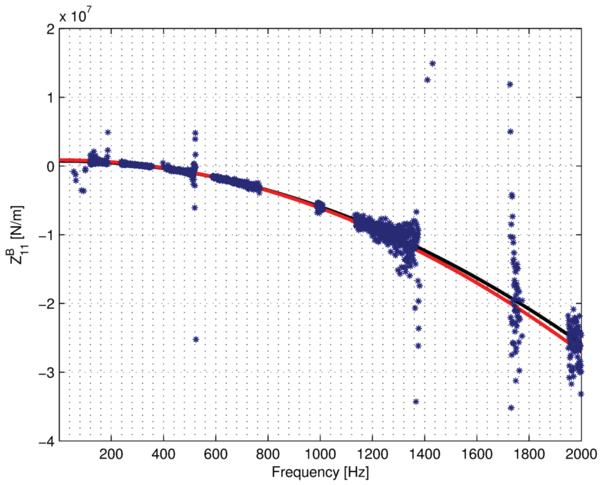

Figure 6: Z_{11}^{B} from formulation 4

3 Analysis of the condition number

From the previous section it is apparent that the best formulation is number 4 (figure 6). Let us try to improve it yet a bit more, trying to find a way to identify the frequencies where the noise is the lowest.

The described iterative process is nothing but an approximate way of determining the solution of a least squares problem [9]. It happens often that an analysis of the condition number of some of the matrices involved is quite important to obtain a good solution. Here, the study of the condition of part of the iteration matrix has been undertaken.

From eq. (21) is clear that both matrices $\boldsymbol{G}$ and $\boldsymbol{P}$ include the term $\left(\boldsymbol{H}_{jj}^{A}-\boldsymbol{H}_{jj}^{C}\right)$, already identified by Ren *et al* [11] as responsible for the instability in the uncoupling process. Thus, a function $r(\omega)$ is defined as

$$r(\omega)=\tau\cdot\ell n(cond(\boldsymbol{H}_{jj}^{A}-\boldsymbol{H}_{jj}^{C})) \tag{23}$$

where the parameter τ is an amplifier of the condition effect, as one can observe in figure 7. The local minima of $r(\omega)$ represent the frequency range where the problem is better conditioned. Therefore, it is advisable to choose only those frequencies in the iterative process; $r(\omega)$ represents a set of points with great dispersion and so, to facilitate the choice of frequencies, a regression of those points has been carried out using the following *MatLab*® function:

$$R(\omega) = \mathrm{csaps}(r(\omega), v) \tag{24}$$

where v is a weighting parameter varying between 0 and 1. If $v = 0$ the regression is linear; $v = 1$ corresponds to an interpolation passing through all the points. To determine the local minima one chose the frequency range where the derivative of $R(\omega)$ is less than a parameter ε and the second derivative is positive:

$$\frac{dR(\omega)}{d\omega} < \varepsilon \qquad \wedge \qquad \frac{d^2R(\omega)}{d\omega^2} > 0 \tag{25}$$

3.1 Simulation study

To illustrate this procedure let us apply it to the problem described in section 2.2 (also with 3% added noise). The chosen parameters are: $\tau = 30$, $v = 10^{-6}$ e $\varepsilon = 0{,}05$. Figure 7 represents $r(\omega)$, $R(\omega)$ and the chosen frequencies, identified by the vertical grey bands.

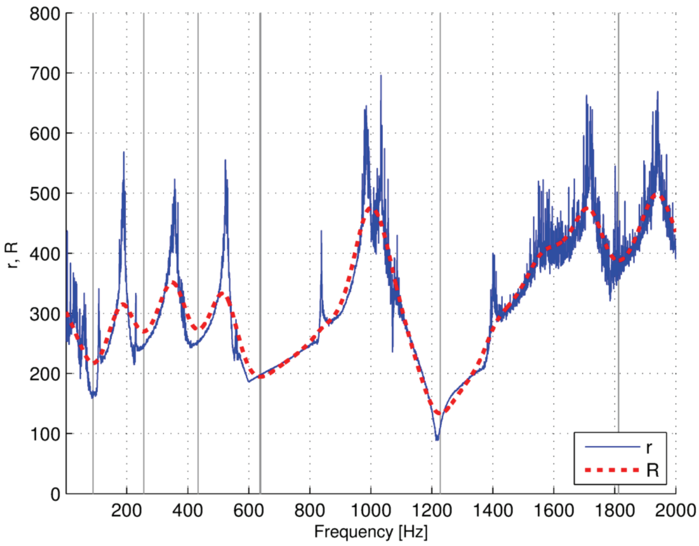

Figure 7: $r(\omega)$ and $R(\omega)$ of $\left(\boldsymbol{H}_{jj}^{A} - \boldsymbol{H}_{jj}^{C}\right)$

Taking figure 6 and superimposing the chosen frequency bands, one can see (figure 8) that those bands coincide with some zones where the scatter of points is smaller. In contrast, the frequency ranges where $r(\omega)$ has local maxima (figura 7) coincide with the frequencies where there is higher vertical perturbation (figure 8).

Taking only the frequencies defined by the vertical grey bands, a new fitting has been performed, obtaining a much better match for the dynamic stiffness, as the red curve now coincides with the black one, as shown in figure 9.

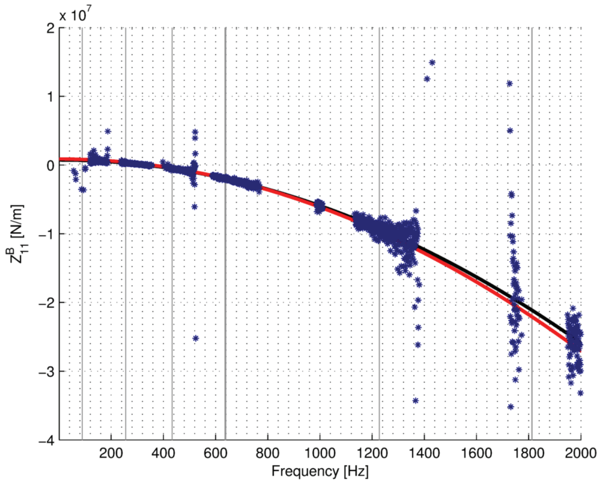

Figure 8: Z_{11}^B from formulation 4, with the indication of the frequency bands with minimum condition values

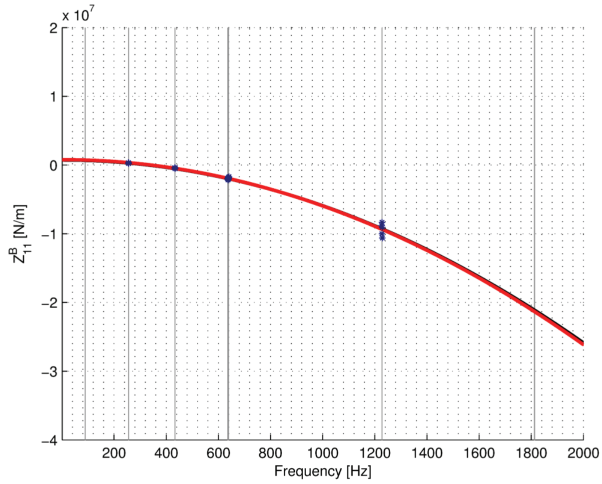

Figure 9: Z_{11}^B from formulation 4, using the frequency bands with minimum condition values

After the dynamic stiffness has been evaluated, with the calculation of the mass and stiffness matrices of substructure *B*, the receptance $\boldsymbol{H}^B = (\boldsymbol{K}^B - \omega^2 \boldsymbol{M}^B)^{-1}$ has been calculated. Figure 10 shows in the blue line the receptance $\boldsymbol{H}_{11}^B$ when all the blue points of figure 8 are considered and, in red, when only the blue points of figure 9 are taken; the black line represents the exact solution. Figure 11 shows similar results for the receptance $\boldsymbol{H}_{12}^B$. In both cases it is obvious the improvement in the results when the criterion of minimum condition number is applied. However, a spurious resonance at low frequency still remains.

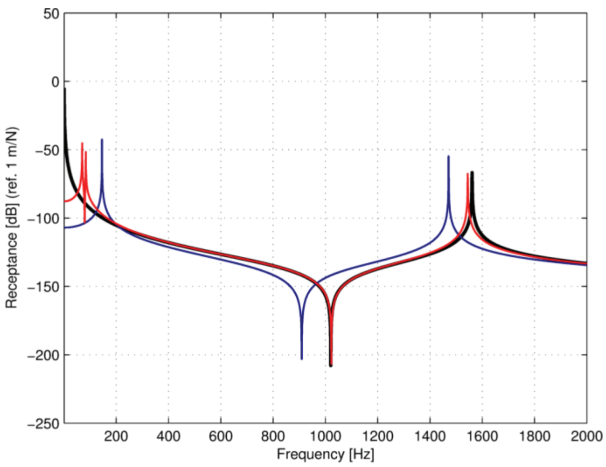

Figure 10: H_{11}^B from formulation 4; blue line: with all the points of fig. 8; red line: with all the points of fig. 9; black line: exact solution

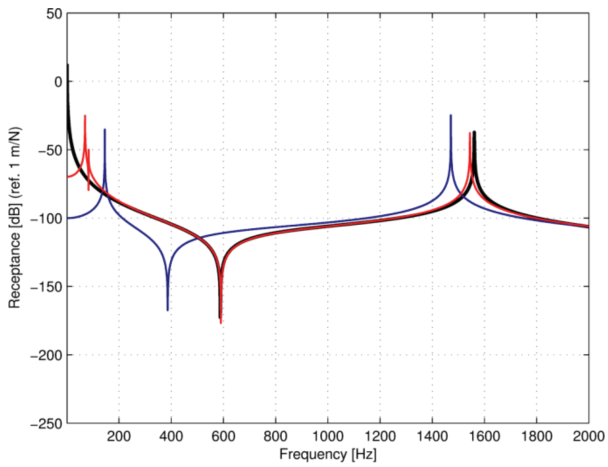

Figure 11: H_{12}^B from formulation 4, following a similar procedure as for H_{11}^B of figure 10

In the next section one shall apply the same procedure to another uncoupling technique, the one developed in Refs. [3, 12].

4 Ren's uncoupling technique

In the works of Ren [3, 12], the substructure B is characterized through its dynamic stiffness, using the force equilibrium of two coupled substructures. The methodology that he followed is different from the one that is presented next, which is somewhat simpler, leading to the same results.

Starting with eq. (1), this can be written as:

$$\boldsymbol{Z}^{C} = \left(\boldsymbol{I} + \boldsymbol{Z}^{B}\boldsymbol{H}^{A}\right)\boldsymbol{Z}^{A} \tag{26}$$

Inverting both sides of eq. (26),

$$\boldsymbol{H}^{C} = \boldsymbol{H}^{A}\left(\boldsymbol{I} + \boldsymbol{Z}^{B}\boldsymbol{H}^{A}\right)^{-1} \tag{27}$$

Post-multiplying eq. (27) by $\left(\boldsymbol{I} + \boldsymbol{Z}^{B}\boldsymbol{H}^{A}\right)$, leads to

$$\boldsymbol{H}^{C} + \boldsymbol{H}^{C}\boldsymbol{Z}^{B}\boldsymbol{H}^{A} = \boldsymbol{H}^{A} \tag{28}$$

In explicit terms, with co-ordinates *i* and *j*, it follows that

$$\begin{bmatrix} \boldsymbol{H}_{ii}^{C} & \boldsymbol{H}_{ij}^{C} \\ \boldsymbol{H}_{ji}^{C} & \boldsymbol{H}_{jj}^{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Z}_{jj}^{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{H}_{ii}^{A} & \boldsymbol{H}_{ij}^{A} \\ \boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_{ii}^{A} & \boldsymbol{H}_{ij}^{A} \\ \boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A} \end{bmatrix} - \begin{bmatrix} \boldsymbol{H}_{ii}^{C} & \boldsymbol{H}_{ij}^{C} \\ \boldsymbol{H}_{ji}^{C} & \boldsymbol{H}_{jj}^{C} \end{bmatrix} \tag{29}$$

and finally,

$$\begin{bmatrix} \boldsymbol{H}_{ij}^{C}\boldsymbol{Z}_{jj}^{B}\boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{ij}^{C}\boldsymbol{Z}_{jj}^{B}\boldsymbol{H}_{jj}^{A} \\ \boldsymbol{H}_{jj}^{C}\boldsymbol{Z}_{jj}^{B}\boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{C}\boldsymbol{Z}_{jj}^{B}\boldsymbol{H}_{jj}^{A} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_{ii}^{A} & \boldsymbol{H}_{ij}^{A} \\ \boldsymbol{H}_{ji}^{A} & \boldsymbol{H}_{jj}^{A} \end{bmatrix} - \begin{bmatrix} \boldsymbol{H}_{ii}^{C} & \boldsymbol{H}_{ij}^{C} \\ \boldsymbol{H}_{ji}^{C} & \boldsymbol{H}_{jj}^{C} \end{bmatrix} \tag{30}$$

The four equations included in expression (30) are the same as those obtained by Ren [3]. Let us name each one of them:

Formulation R1

$$\boldsymbol{H}_{ij}^{C}\boldsymbol{Z}_{jj}^{B}\boldsymbol{H}_{ji}^{A} = \boldsymbol{H}_{ii}^{A} - \boldsymbol{H}_{ii}^{C} \tag{31}$$

Formulation R2

$$\boldsymbol{H}_{ij}^{C}\boldsymbol{Z}_{jj}^{B}\boldsymbol{H}_{jj}^{A} = \boldsymbol{H}_{ij}^{A} - \boldsymbol{H}_{ij}^{C} \tag{32}$$

Formulation R3

$$\boldsymbol{H}_{jj}^{C}\boldsymbol{Z}_{jj}^{B}\boldsymbol{H}_{ji}^{A} = \boldsymbol{H}_{ji}^{A} - \boldsymbol{H}_{ji}^{C} \tag{33}$$

Formulation R4

$$\boldsymbol{H}_{jj}^{C}\boldsymbol{Z}_{jj}^{B}\boldsymbol{H}_{jj}^{A} = \boldsymbol{H}_{jj}^{A} - \boldsymbol{H}_{jj}^{C} \tag{34}$$

4.1 Algorithm of the joint

Ren [3] presented a linear form of resolution for each of the four cases. Each formulation has the following format:

$$\boldsymbol{H}_{kj}^{C}\boldsymbol{Z}_{jj}^{B}\boldsymbol{H}_{jh}^{A} = \boldsymbol{H}_{kh}^{AC} \tag{35}$$

If $k \geq j$ and $h \geq j$, which happens almost every time, as normally one has more co-ordinates outside the joint than at the joint itself, one can rewrite eq. (35) and obtain a liner system of equations of the following type:

$$\boldsymbol{E}_{pq}\boldsymbol{z}_{q} = \boldsymbol{b}_{p} \tag{36}$$

where $\boldsymbol{E}_{pq}$, $\boldsymbol{z}_{q}$ and $\boldsymbol{b}_{p}$ are given by

$$\boldsymbol{E}_{pq} = \boldsymbol{H}_{dn}^{C}\boldsymbol{H}_{me}^{A} \qquad \boldsymbol{z}_{q} = \boldsymbol{Z}_{nm}^{B} \qquad \boldsymbol{b}_{p} = \boldsymbol{H}_{de}^{AC} \tag{37}$$

and

$$\begin{matrix} p = (d-1)h + e & d = 1,\ldots,k & m = 1,\ldots,j \\ q = (n-1)j + m & n = 1,\ldots,j & e = 1,\ldots,h \end{matrix} \tag{38}$$

Ren [3] suggests the decomposition of the vector of unknowns $\boldsymbol{z}$ so that it becomes independent of the frequency. In order to compare Ren's results with our iterative method and be able to apply the frequency choice procedure described in section 3, one shall simply solve eq. (36) in a least-squares sense to obtain directly the dynamic stiffness of *B*.

4.2 Simulation study

The problem described in section 2.2 is used once again. As it happened in section 2.2, one verifies from the observation of figures 12 to 15 that formulation R4 – the one that uses only the co-ordinates of the joint – is the one that presents the best regression results. In fact, the blue points in figure 15 have lesser dispersion than in figures 12 to 14. As in figure 15 the vertical perturbations tend to appear at the same frequencies as in figure 6, one shall apply the same procedure as before, choosing the frequencies where the condition number is smaller.

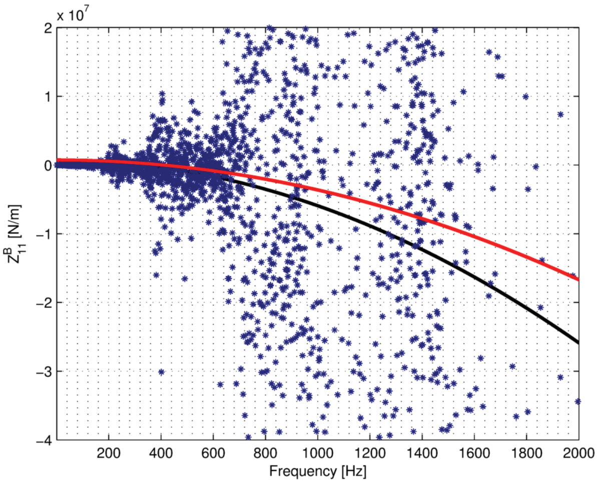

Figure 12: Z_{11}^{B} from formulation R1

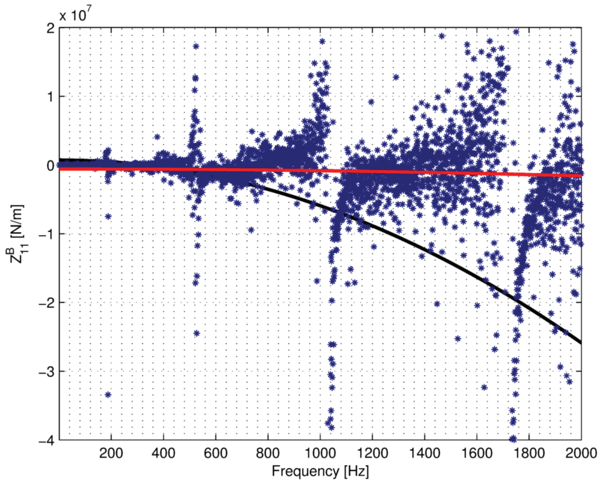

Figure 13: Z_{11}^{B} from formulation R2

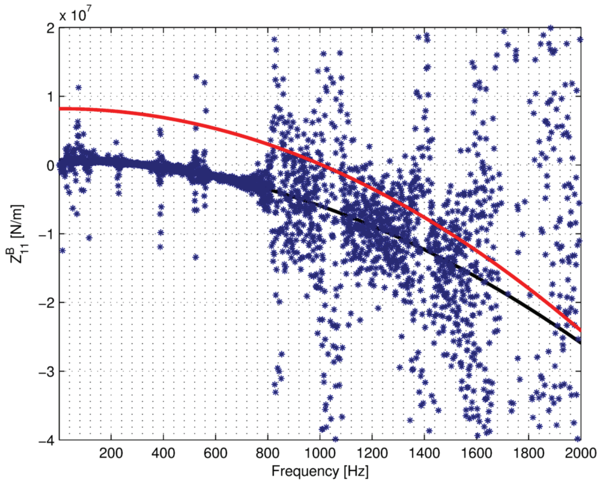

Figure 14: Z_{11}^{B} from formulation R3

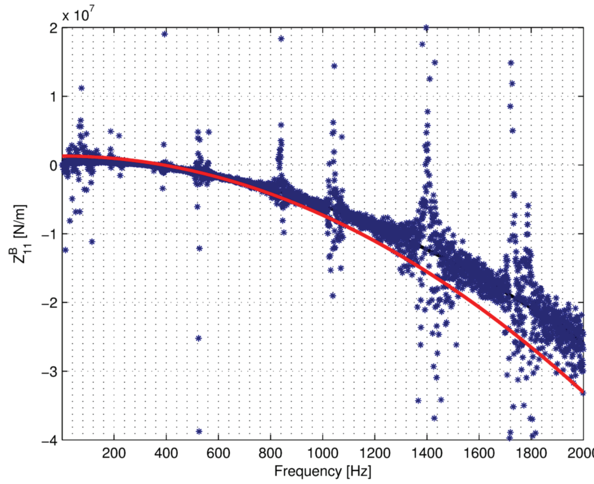

Figure 15: Z_{11}^{B} from formulation R4

In fact, both formulations 4 (eq. (21) and R4 (eq. (34)) include the same matrix difference $\left(\boldsymbol{H}_{jj}^{A}-\boldsymbol{H}_{jj}^{C}\right)$. Therefore, one can use the same frequency bands of figure 7. In figure 16 one can confirm that the vertical bands that identify the chosen frequencies coincide with the zones where the perturbations are smaller, as it also happened in figure 8.

With the chosen frequencies a new fitting was performed and a much better approximation for the dynamic stiffness could be achieved (figure 17). In this case the improvement is even more effective than the one observed between figures 8 and 9 (mostly because the results in figure 8 are already much better than those of figure 16).

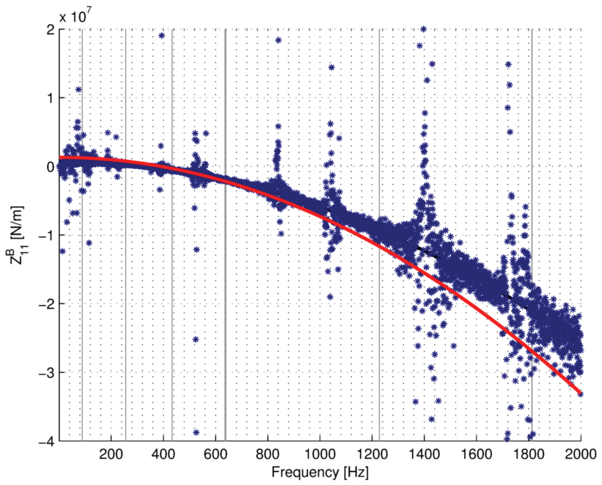

Figure 16: Z_{11}^{B} from formulation R4, with the indication of the frequency bands with minimum condition values

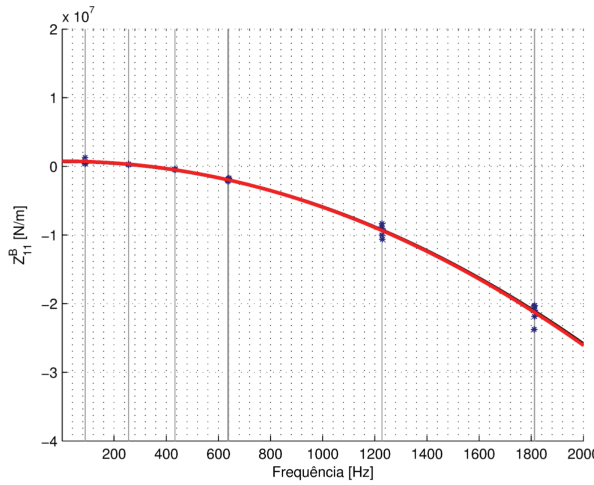

Figure 17: Z_{11}^{B} from formulation R4, using the frequency bands with minimum condition values

In figures 18 and 19 two receptances of *B* are shown, in the same way as those presented in figures 10 and 11. As before, the results improve with the inclusion of the information about the condition of the matrix. However, the results are not so satisfactory as those of figures 10 and 11, obtained with the iterative method.

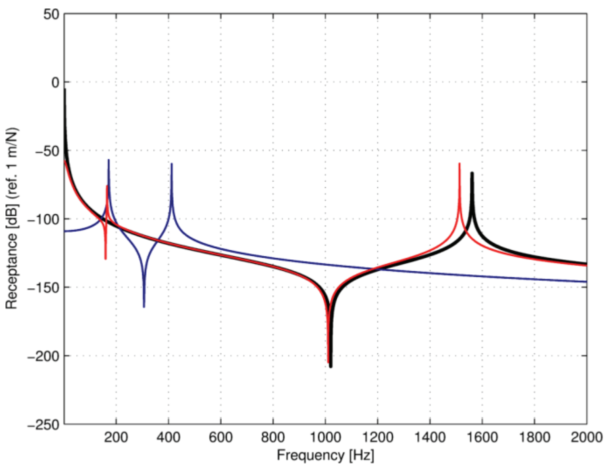

Figure 18: H_{11}^{B} from formulation R4; blue line: with all the points of fig. 16; red line: with all the points of fig. 17; black line: exact solution

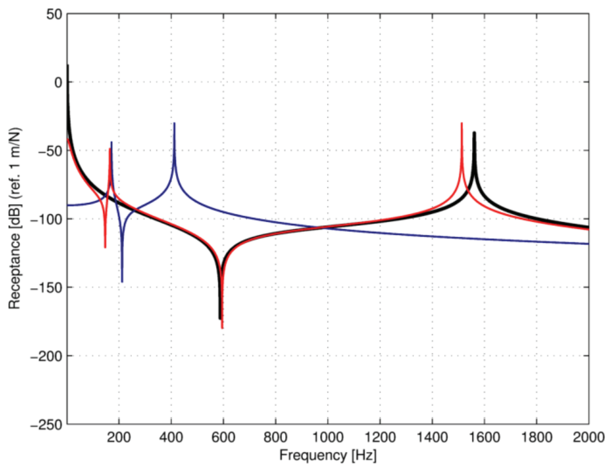

Figure 19: H_{12}^{B} from formulation R4, following a similar procedure as for H_{11}^{B} of figure 18

5 Conclusions

A new iterative method for the identification of joints has been presented; the technique does not need to invert any matrices, though it requires the responses at the joint co-ordinates. Batista and Maia [13], using the classic uncoupling procedure of Jetmundsen *et al* [2] had also concluded that it is necessary to have information at the joint co-ordinates for the uncoupling.

To improve the results, the condition of the matrix representing the difference between the dynamic responses of substructure *A* and structure *C* has been assessed. The identification of the frequencies where the condition number is a minimum revealed itself as very effective, leading to a significant improvement in the uncoupling results. The same procedure has also been applied successfully to the well known Ren's technique. In terms of comparison, it was concluded that the iterative technique led to more accurate results.

The experimental implementation of the proposed method has still to be investigated, as there are some problems, like the difficulty in measuring rotational degrees of freedom. However, these may be estimated using a recently proposed technique [14].

Acknowledgements

This work was partially supported by Portuguese Foundation of Science and Technology (FCT) under the project POCI 2010 and the PhD grant SFRH/BD/29896/2006.

References

[1] Klerk, D., Rixen, D. J., Voormeeren, S. N., *General Framework for Dynamic Substructuring: History, Review, and Classification of Techniques*, Journal of the American Institute of Aeronautics and Astronautics (AIAA), Vol. 46 (5), (2008), pp. 1169-1181.

[2] Jetmundsen, B., Bielawa, R., Flannelly, W. G., *Generalized Frequency Domain Substruture Synthesis*, Jounal of the American Helicopter Society, Vol. 33 (1), (1988), pp. 55-64.

[3] Ren, Y. *The Analysis and Identification of Friction Joint Parameters in the Dynamic Response of Structures*, Ph.D. Thesis, Mechanical Engineering Department, Imperial College of Science and Technology, University of London, U.K., (1992).

[4] Liu, W., Ewins, D. J., *Substructure Synthesis via Elastic Media, Part I: Joint Identification*, Proceedings of the 18th International Modal Analysis Conference, San Antonio, USA, (2000).

[5] Ratcliffe, J. M., Lieven, N. A., *A Generic Element-Based Method for Joint Identification*, Mechanical Systems and Signal Processing, Vol. 4 (1), (1998), pp. 3-28.

[6] Wang, J. H., Chuang, S. C., *Reducing Errors in the Identification of Structural Joint Parameters Using Error Functions*, Journal of Sound and Vibration, Vol. 273 (1,2), (2004), pp. 295-316.

[7] Ren, Y., Beards, C. F., *An Iterative FRF Joint Identification Technique*, Proceedings of the 11th International Modal Analysis Conference, Vol. 2, Kissimmee, Florida, USA, (1993), pp. 1133-1139.

[8] Axelsson, O., *Iterative Solution Methods*, Cambridge University Press, Cambridge, UK, (1996).

[9] Chen, Y.-L., Tan, X.-Y., *Computing Generalized Inverses of Matrices by Iterative Methods Based on Splittings of Matrices*, Applied Mathematics and Computation, Vol. 163 (1), (2005), pp. 309 – 325.

[10] Liu, W., *Structural Dynamics Analysis and Testing of Coupled Structures,* Ph.D. Thesis, Mechanical Engineering Department, Imperial College of Science and Technology, University of London, U.K., (2000).

[11] Ren, Y., Beards, C. F., *On the Importance of Weighting on FRF Joint Identification*, Proceedings of the 11th International Modal Analysis Conference, Vol. 2, Kissimmee, Florida, USA, (1993), pp. 1606-1611.

[12] Ren, Y., Beards, C. F., *Identification of Joint Properties of a Structure Using Frf Data*, Journal of Sound and Vibration, Vol. 186, (4), (1995), pp. 567-587.

[13] Batista, F. C., Maia, N. M. M., *Uncoupling Techniques for the Dynamic Characterization of Sub-structures*, Linking Models and Experiments, Vol. 2, Conference Proceedings of the Society for Experimental Mechanics Series, Vol. 4, DOI: 10.1007/978-1-4419-9305-2_28, (2011), pp. 383-392.

[14] Batista, F. C., Maia, N. M. M., *Estimation of Unmeasured Frequency Response Functions*, International Congress on Sound and Vibration (ICSV 19), Vilnius, Lithuania (2012).